Problem 2.4

The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. **(a)** Assuming the projectile has a cross-sectional area A (normal to its velocity) and speed v, and that the density of the fluid is ϱ, show that the rate at which the projectile encounters fluid (mass/time) is ϱAv. **(b)** Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is ϱAv^2. It is certainly not true that all the fluid that the projectile encounters is accelerated to the full speed v, but one might guess that the actual force would have the form

$$f_{\text{quad}} = \kappa \varrho A v^2 \tag{2.84}$$

where κ is a number less than 1, which would depend on the shape of the projectile, with κ small for a streamlined body, and larger for a body with a flat front end. This proves to be true, and for a sphere the factor κ is found to be $\kappa = 1/4$. **(c)** Show that (2.84) reproduces the form (2.3) for f_{quad}, with c given by (2.4) as $c = \gamma D^2$. Given that the density of air at STP is $\varrho = 1.29\ \text{kg/m}^3$ and that $\kappa = 1/4$ for a sphere, verify the value of γ given in (2.6).

Solution

Part (a)

Consider a projectile moving in one dimension through still air at a constant velocity v. Imagine for the moment that the projectile consists of a cross-sectional area through which air can pass. The rate that air passes through this area is dm/dt.

$$\frac{dm}{dt} = \frac{d}{dt}(m)$$

Mass is density times volume. This is the air density, and the volume is the amount of space that gets swept through in the infinitesimal time the projectile has to move.

$$\frac{dm}{dt} = \frac{d}{dt}(\varrho V)$$

Volume is cross-sectional area times distance.

$$\frac{dm}{dt} = \frac{d}{dt}(\varrho A x)$$

Bring the constants in front.

$$\frac{dm}{dt} = \varrho A \frac{d}{dt}(x)$$

The rate of change of distance with respect to time is the projectile's velocity.

$$\frac{dm}{dt} = \varrho A v$$

Part (b)

Begin with the result of part (a).

$$\frac{dm}{dt} = \varrho A v$$

Multiply both sides by v.

$$v\frac{dm}{dt} = \varrho A v^2$$

The velocity that the projectile is travelling at is a constant, so it can be brought inside the derivative.

$$\frac{d}{dt}(mv) = \varrho A v^2$$

The product of mass and velocity is momentum.

$$\frac{dp}{dt} = \varrho A v^2$$

By Newton's second law, the rate of change of momentum is the sum of the forces in the direction the projectile is moving in.

$$F_{\text{net}} = \varrho A v^2$$

This is the net force of the projectile on the air, assuming the air does not pass through the projectile but rather collides with it and accelerates to its velocity. Therefore, by Newton's third law, this is the net drag force on the projectile.

Part (c)

The aim is to show that the given expression for f_{quad},

$$f_{\text{quad}} = \kappa \varrho A v^2, \tag{2.84}$$

simplifies to

$$f_{\text{quad}} = cv^2, \tag{2.3}$$

where $c = \gamma D^2$ for spherical projectiles and $\gamma = 0.25\ \text{N}\cdot\text{s}^2/\text{m}^4$ for projectiles in air at STP.

$$\begin{aligned} f_{\text{quad}} &= \kappa \varrho A v^2 \\ &= \underbrace{(\kappa \varrho A)}_{=\gamma D^2} v^2 \\ &= \gamma D^2 v^2 \\ &= cv^2 \end{aligned}$$

Check to see that this relationship between κ and γ yields the right value for γ. For a spherical projectile, the cross-section is that of a circle with area $A = \pi r^2 = \pi(D/2)^2 = \pi D^2/4$.

$$\gamma D^2 \stackrel{?}{=} \kappa \varrho A$$

$$\stackrel{?}{=} \kappa \varrho \left(\frac{1}{4}\pi D^2\right)$$

Divide both sides by D^2.

$$\gamma \stackrel{?}{=} \frac{1}{4}\pi\kappa\varrho$$

$$\stackrel{?}{=} \frac{1}{4}\pi\left(\frac{1}{4}\right)\left(1.29\ \frac{\text{kg}}{\text{m}^3}\right)$$

$$\approx 0.25\ \frac{\text{N}\cdot\text{s}^2}{\text{m}^4}$$

It checks out.